Distortion in Voting

Soumyajit Pyne

29 October, 2026

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a	b
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Revealing Utility

$u_1(a) = 1$, $u_1(b) = 0$

$u_2(a) = 1/2$, $u_2(b) = 1/2$

→ Consistent with the ranking

We should choose 'a'.

Formalising the concept

- A set of n voters V and a set of m candidates C.
- Each voter $i \in V$ has a strict ranking $\succ_i$ over the candidates in C.
- An input instance $I = (V, C, \succ)$

Formalising the concept

- For each voter $i \in V$, there is a utility function $u_i : C \to \mathbb{R}_{\geq 0}$.
- Given an instance $I = (V, C, \succ)$, and a utility profile $U = \{u_1, u_2, \ldots u_n\}$, U is consistent with I if for each $i \in V$ and $j, j' \in C$,

$$j \succ_i j' \Rightarrow u_i(j) \geq u_i(j')$$

Formalising the concept

(*) Given an instance I and a consistent utility profile V, define,

$$SW(j, V) = \sum_{i \in V} v_i(j)$$

Formalising the concept

- Given an instance I and a consistent utility profile U, define,

$$sw(j, U) = \sum_{i \in V} U_i(j)$$

- A voting rule f takes an input instance $I = (V, C, \succ)$ and outputs a candidate from C.

$$dist(f) = \sup_{I = (V, C, \succ)} \max_{j \in C,\ U:\ U \text{ consistent with } I} \frac{sw(j, U)}{sw(f(I), U)}$$

Randomised voting rules

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v_1	v_2
$a(1)$	$b(M)$
$b(0)$	$a(0)$

$dist(f) \geq M$ for any $M > 0$.

Unit sum utilities

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$sw(a, U) \geq 1/2$

$sw(b, U) \leq 1 + 1/2 = 3/2$

Plurality has high distortion

* $n = 10$, $m = 3$

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	Plurality chooses 'c'
a	a	a	b	b	b	c	c	c	c	
b	b	b	a	a	a	b	b	b	b	
c	c	c	c	c	c	a	a	a	a	

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c	c	c	c	c	c	a	a	a	a

Plurality chooses 'c'

gives ½ to the first two ranks

gives ⅓ to all

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b	b	b	a	a	a	b	b	b	b	
c	c	c	c	c	c	a	a	a	a	

gives $1/2$ to the first two ranks

gives $1/3$ to all

$sw(c) = 4/3$, $sw(b) = 1/2 \times 6 + 1/3 \times 4 = 3 + 4/3$

$dist(\text{plurality}) \geq 13/4 > 3 = m$

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c	c	c	c	c	c	a	a	a	a

Plurality chooses 'c'

gives $1/2$ to the first two ranks — gives $1/3$ to all

$sw(c) = 4/3$, $sw(b) = \frac{1}{2} \times 6 + \frac{1}{3} \times 4 = 3 + 4/3$

$dist(\text{plurality}) \geq 13/4 > 3 = m$

H.w.: Show that plurality has dist. of $\Theta(m)$

Trivial upper bound on distortion

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$$dist(f) \leq \frac{n}{n/m} = m$$

Almost tight lower and upper bound on distortion

Thm 1. There is a randomized voting rule that achieves a distortion at most $\sqrt{m \log m}$.

Thm 2. No randomized voting rule can achieve a distortion lesser than $\sqrt{m}$.

Introduction to metric distortion

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(i) $d(x,y) \geq 0$ $\forall x, y$ (ii) $d(x,y) \leq d(x,z) + d(y,z)$ (triangle inequality)

(iii) $d(x,y) = d(y,x)$ $\forall x, y$ (iv) $d(x,x) = 0$ $\forall x$

Introduction to metric distortion

* Given an instance $I = (V, C, >)$, a function $d : \widetilde{(V \cup C)} \longrightarrow \mathbb{R}_{\geq 0}$ is a metric and consistent with I if

(i) $d(x,y) \geq 0$ $\forall x, y$ (ii) $d(x,y) \leq d(x,z) + d(y,z)$ (triangle inequality) $\forall x$

(iii) $d(x,y) = d(y,x)$ $\forall x, y$ (iv) $d(x,x) = 0$

$\forall i \in V$ and $j, j' \in C$, $j >_i j' \Rightarrow d(i,j) \leq d(i,j')$

Metric distortion

* Given an instance $I = (V, C, >)$ and a metric d consistent with I, define,

$$\forall j \in C, \quad cost(j, d) = \sum_{i \in V} d(i, j)$$

$$dist(f) = \sup_{I} \max_{j \in C,\ d:\ d \text{ consistent with } I} \frac{cost(f(I), d)}{cost(j, d)}$$

Lower bound on deterministic voting rules

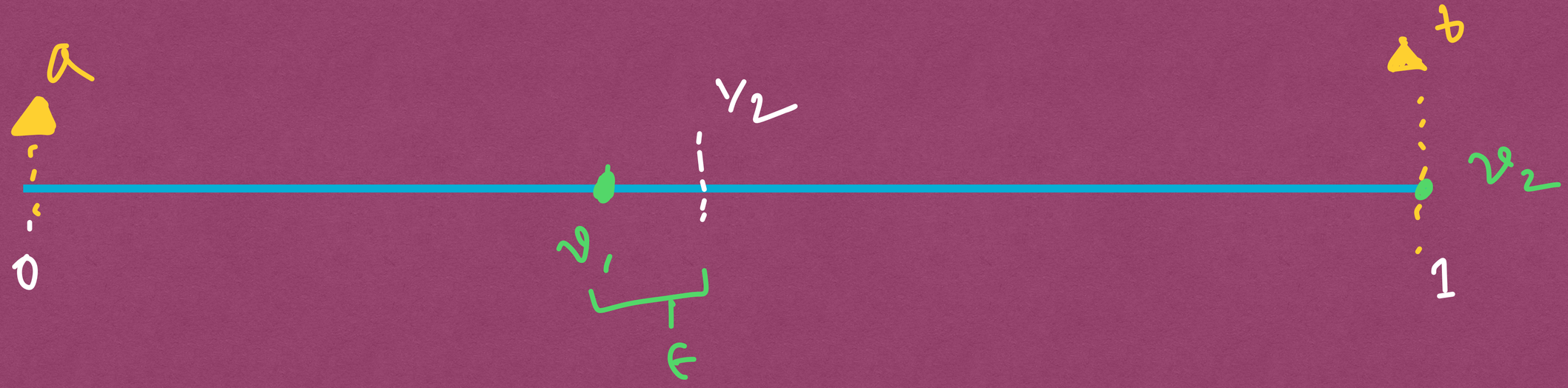
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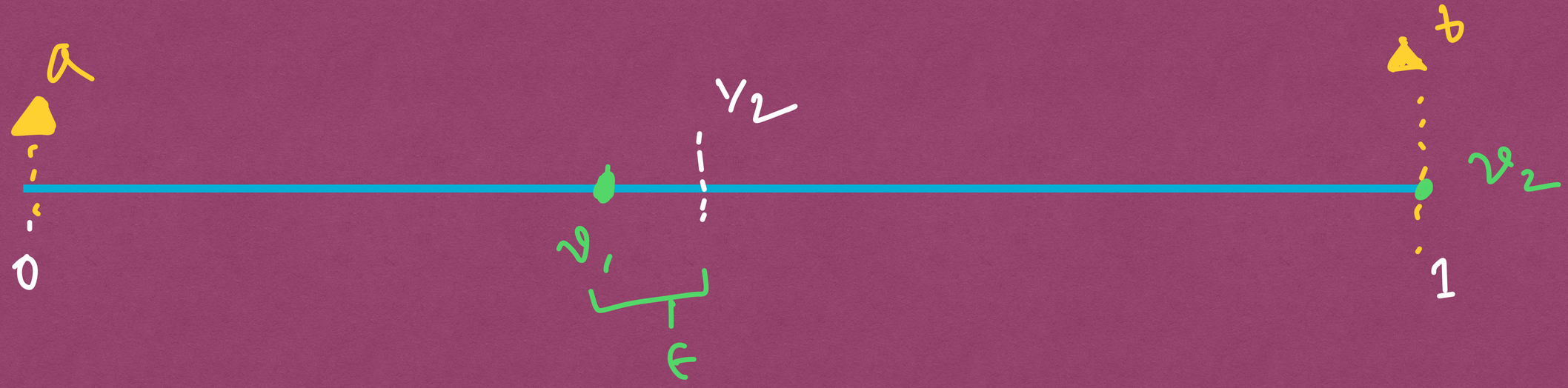
Lower bound on deterministic voting rules

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W.log. f chooses a

$cost(a) = 1 + Y_2 - \epsilon = 3/2 - \epsilon$

$cost(b) = Y_2 + \epsilon$, $\lim_{\epsilon \to 0}$ $dist_\epsilon = 3$



Upper bound on deterministic voting rules

Plurality Veto:

1. $\text{Plu}_j \leftarrow$ plurality score of $j \in C$
2. For $i = 1 \ldots n-1$:
3. $\quad j_i \leftarrow$ least favourite candidate j of i with $\text{plu}_j > 0$
4. $\quad \text{plu}_j \leftarrow \text{plu}_j - 1$
5. Return the candidate j with $\text{plu}_j > 0$.

Thm 3: Plurality veto achieves a distortion of 3.